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Solution for 'Topics in complex analysis'

(08/10/2025)

H 5.1 (Absolute convergence of infinite products)
Let $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ be a sequence. We say that the infinite product $\prod_{j=1}^{\infty} a_j$ converges absolutely if there exists $j_0 \in \mathbb{N}$ such that $a_j \notin (-\infty, 0]$ for all $j \geq j_0$ and $\sum_{j\geq j_0} |\log(a_j)| < +\infty$.

a) Show that the absolute convergence of an infinite product implies its convergence.

b) Show that an infinite product of the form $\prod_{j=1}^{\infty}(1+b_j)$ converges absolutely if and only if $\sum_{j=1}^{\infty} |b_j| < +\infty$.

Solution H 5.1:

a) This statement is an immediate consequence of Lemma 3.3, since the convergence of $\sum_{j\geq j_0} |\log(a_j)|$ implies convergence of $\sum_{j\geq j_0} \log(a_j)$.

b) Since $\log(1) = 0$ and $\log'(1) = 1$ it follows that for $|z|$ small enough we have

$$\frac{1}{2}|z| \leq |\log(1+z)| \leq \frac{3}{2}|z|.$$

Since the convergence of either $\sum_{j=1}^{\infty} |b_j|$ or $\prod_{j=1}^{\infty}(1+b_j)$ implies that $b_j \to 0$ as $j \to +\infty$, we conclude that in either case there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ we have

$$\frac{1}{2}|b_j| \leq |\log(1+b_j)| \leq \frac{3}{2}|b_j|.$$

Thus the claim follows from the definition of absolute convergence of infinite products. □

H 5.2 (Examples of infinite products)
Examine if the following infinite products exist in the sense of Definition 3.1. If so, calculate their value.

a) $\prod_{n=1}^{\infty}\left(1 - \frac{1}{(n+1)^2}\right)$ b) $\prod_{n=1}^{\infty}\left(1 - \frac{1}{n}\right)$ c) $\prod_{n=3}^{\infty} \frac{n^2-4}{n^2-1}$ d) $\prod_{n=1}^{\infty} \frac{(1+n^{-1})^2}{1+2n^{-1}}$

Hint: In all examples you can directly calculate the value of the partial products.

Solution H 5.2:
In all four examples only finitely many factors equal zero. Hence we only have to treat the convergence of the non-zero factors.

a) Set $a_n = \left(1 - \frac{1}{(n+1)^2}\right)$. Then $a_n = \frac{n(n+2)}{(n+1)(n+1)}$. Thus by the telescopic structure the finite product $\prod_{n=1}^m a_n$ simplifies to

$$\prod_{n=1}^{m} a_n = \frac{1}{m+1} \cdot \frac{m+2}{2}.$$

Hence $\lim_{m\to+\infty} \prod_{n=1}^{m} a_n = \frac{1}{2}$, so that by definition the product converges.

b) Set $a_n = \left(1 - \frac{1}{n}\right)$. Then only $a_1 = 0$ and in general $a_n = \frac{n-1}{n}$. Again by the telescopic structure the finite product $\prod_{n=2}^m a_n$ reduces to

$$\prod_{n=2}^{m} a_n = \frac{1}{m}.$$

Hence $\lim_{m\to+\infty} \prod_{n=2}^{m} a_n = 0$, so that by definition the product does not converge.

c) Set $a_n = \frac{n^2-4}{n^2-1}$. Then $a_n \neq 0$ for all $n \geq 3$. Moreover, $a_n = \frac{(n+2)(n-2)}{(n+1)(n-1)}$. Thus by the telescopic structure the finite product $\prod_{n=3}^m a_n$ simplifies to

$$\prod_{n=3}^{m} a_n = \frac{m+2}{4} \cdot \frac{1}{m-1}.$$

Hence $\lim_{m\to+\infty} \prod_{n=3}^{m} a_n = \frac{1}{4}$, so that by definition the product converges.

d) Set $a_n = \frac{(1+n^{-1})^2}{1+2n^{-1}}$. Then $a_n = \frac{(n+1)(n+1)}{n(n+2)} \neq 0$. The telescopic structure implies that

$$\prod_{n=1}^{m} a_n = \frac{m+1}{1} \cdot \frac{2}{m+2}.$$

Hence $\lim_{m\to+\infty} \prod_{n=1}^{m} a_n = 2$, so that by definition the product converges. □

H 5.3 (A class of diverging products)

Let $\{a_j\}_{j\in\mathbb{N}} \subset [0, +\infty)$ be a sequence such that $\sum_{j=1}^{\infty} 1 - a_j = +\infty$. Show that $\lim_{n\to\infty} \prod_{j=1}^{n} a_j = 0$.

Hint: Use that $t \leq \exp(t-1)$ for all $t \in \mathbb{R}$.

Solution H 5.3:

Since $t \leq \exp(t-1)$ for all $t \in \mathbb{R}$ we deduce that

$$0 \leq \prod_{j=1}^{n} a_j \leq \prod_{j=1}^{n} \exp(a_j - 1) = \exp\Big(-\sum_{j=1}^{n} 1 - a_j\Big).$$

Using the assumption, the claim follows after letting $n \to +\infty$. □

H 5.4 (A useful criterion for the convergence of infinite products)

Let $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$. Assume that $\sum_{j=1}^{\infty} |a_j|^2 < +\infty$. Show that $\prod_{j=1}^{\infty}(1 + a_j)$ converges if and only if $\sum_{j=1}^{\infty} a_j$ converges. Conclude that the infinite product $\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)$ converges only for $z = 0$.

Solution H 5.4:
The assumption $\sum_{j=1}^{\infty} |a_j|^2 < +\infty$ implies that there exists $j_0 \in \mathbb{N}$ such that $|a_j| \leq \frac{1}{2}$ for all $j \geq j_0$. Let us set

$$\theta_j = \frac{\log(1+a_j) - a_j}{a_j^2} = \sum_{n=2}^{\infty} (-1)^{n+1} \frac{a_j^{n-2}}{n}.$$

Since $a_j \to 0$ as $j \to +\infty$, it follows that $\lim_{j\to+\infty} \theta_j = -\frac{1}{2}$. In particular, the sequence θ_j is bounded. Hence the series

$$\sum_{j=j_0}^{\infty} \theta_j a_j^2 = \sum_{j=j_0}^{\infty} (\log(1+a_j) - a_j) \tag{1}$$

converges. Thus on the one hand the convergence of $\sum_{j=1}^{\infty} a_j$ implies that $\sum_{j=j_0}^{\infty} \log(1+a_j)$ converges. Taking the exponential we infer that in this case also $\prod_{j=j_0}^{\infty}(1+a_j)$ converges to a non-zero limit. Thus $\prod_{j=1}^{\infty}(1+a_j)$ converges. On the other hand the convergence of the latter product implies that for a suitably large $j_1 \geq j_0$ it holds that

$$\log\Big(\prod_{j=j_1}^{\infty}(1+a_j)\Big) = \sum_{j=j_1}^{\infty} \log(1+a_j),$$

as in the proof of Lemma 3.3. Hence due to (1) the convergence of the product implies also the convergence of $\sum_{j=j_1}^{\infty} a_j$, which yields the claim.
Applying this result to the product $\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)$ (note that the assumptions are satisfied for all $z \in \mathbb{C}$ fixed) we see that it converges if and only if $\sum_{j=1}^{\infty} \frac{z}{j}$ converges. This is only the case for $z = 0$. □